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UNIAXIAL SHOCK WAVE PROPAGATION IN A VISCOELASTIC MATERIAL

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Abstract—The propagation of shock waves into semi-infinite viscoelastic rods is studied. The particle velocity behind the wave front is expanded in a Taylor series about the front. Conditions under which a wave front is stable to small disturbances are discussed; and, when the front is stable, it is shown how the theory of propagating surfaces of discontinuity can be used to find the partial derivative terms and the time of arrival function appearing in the series when the material ahead of the front is undisturbed. The series is valid as long as the first order wave front remains stable and as long as other shock waves do not form behind the original one. A single integral, nonlinear constitutive equation, which characterizes the response of many elastomers, is used to illustrate the application of the theory.

1. INTRODUCTION

QUITE a number of investigators have looked at wave propagation in nonlinear viscoelastic materials,† but in most cases attempts were not made to find general closed form solutions. One exception to this is the work of Achenbach and Reddy [2], who used a wave front Taylor expansion, in conjunction with the theory of propagating surfaces of discontinuity, to look at acceleration and higher order shear waves propagating into an initially undisturbed nonlinear viscoelastic material characterized by a second order multiple integral constitutive equation.

Among those who have looked at shock waves‡ in viscoelastic materials are Coleman *et al.* [3], Achenbach *et al.* [6], Coleman and Gurtin [7], Pipkin [4], Greenberg [8] and Waterston [9]. Although solutions are presented in [4, 8], they are only for steady motion in media of infinite extent; boundary conditions necessary to generate the rigid wave form solutions found are, of course, not mentioned. In none of the references [3, 4, 6–9] is the wave front expansion approach of [1, 2] used; and, in contrast to shock studies [3, 4, 6–9], we develop a method of solution for unsteady motion.

In this paper we investigate uniaxial, isothermal shock fronts propagating in rods which, to avoid the problem of reflection, are assumed to be semi-infinite. The rods are made of a material having a constitutive equation that is included in the portion of Schapery's single

† Our designation "nonlinear" does not include the semilinear materials considered by Lubliner and Secor [1].

‡ Although some authors [3, 4] call the first order waves occurring in linear viscoelasticity shock waves, we shall call a first order wave front a shock only when the magnitude of its speed of propagation is greater than the magnitude of the sound speed just ahead of it and less than the magnitude of the sound speed just behind it. This identification agrees with that given in [5] for ideal fluids, and it precludes shock waves from occurring in linear or semilinear materials since first order waves and sound waves in these types of materials have a common speed of propagation.

integral constitutive theory that is described in [10]. The nonlinearity of the material is associated with both the entropy production and free energy and appears in the constitutive equation as a strain dependent stiffening or softening factor but does not alter the time scale. The constitutive equation was selected because it is comparatively simple and yet realistic. We emphasize, though, that uniaxial wave propagation in materials having considerably more general isothermal constitutive equations from Schapery's theory can be analyzed with virtually no additional analytical difficulties. In fact, the method can also be applied when the multiple integral theory of Green and Rivlin [11] and Green *et al.* [12] is used. For this theory, however, the work involved in applying the method increases considerably as the number of multiple integral terms in the expansion increases.

In Section 2 we state the uniaxial assumptions and mention the Lagrangian representation to be used.

Two different definitions of the jump operator are given in Section 3, and in Section 4 it is shown that two particular wave front Taylor expansions lead quite naturally to these different definitions for the jump. One of the expansions developed in Section 4 has been considered by Reiss [13] and is a generalization of the expansions used in [1, 2]. The expansions in [1, 2] are for a constant wave speed and, hence, a predetermined time of arrival function. The time of arrival for a shock wave, however, depends upon the boundary conditions and is not in general known *a priori*.

Propagating discontinuity equations corresponding to kinematic compatibility, balance of momentum, and the constitutive equation are given in Section 5.

In Section 6 we find the Lagrangian speed of both a first order wave and a sound wave; in Section 7 we discuss the conditions under which a first order wave front is stable to small disturbances and use the results of Section 6 to find stability criteria for a first order wave propagating into an initially undisturbed material.

Section 8 is used to develop a method to obtain the unknown functions appearing in a wave front Taylor expansion when the material ahead of the front is undisturbed. For definiteness, it is assumed that the particle velocity at the end of the rod, rather than the stress, is specified. It is found that, for a Neohookean type material, the wave front speed can be found as an explicit function of the jump in the particle velocity. The special case of a constant wave front speed is also discussed, and it is noted that the solution for a shock wave cannot be directly reduced to that for an acceleration wave in a nonlinear material or to that for a first order wave in a linear material.

Limitations of the wave front expansion approach are discussed in Section 9.

2. THE UNIAXIAL ASSUMPTION

The specimen that we shall consider is a cylindrical semi-infinite rod which is homogeneous in its undisturbed (unstressed and unstrained) state. It is assumed that the end of the rod is set in motion at time $t = 0$, and that prior to $t = 0$ the rod is undisturbed. The positive $\bar{X}$ axis of a rectangular cartesian coordinate system $\bar{X}$–$\bar{Y}$–$\bar{Z}$ is aligned along the rod such that, prior to $t = 0$, the origin coincides with the plane of the end of the rod. As is often the case in investigating wave motion in rods, it is assumed that the motion of all particles in the rod is one dimensional so that all cross sections remain plane and do not rotate for all time t. Every cross section is given a label X, the Lagrangian coordinate, that, before the wave front arrives at the cross section, is equal to the Eulerian coordinate x of the section. Since the Lagrangian rather than the Eulerian viewpoint is used, the independent variables are X and t rather than x and t; the displacement u is therefore written as

$$u = x(X, t) - X. \tag{1}$$

3. JUMP OPERATORS

A fundamental idea in the theory of singular surfaces is the jump in a function. For one dimensional wave propagation, the jump can be defined in terms of the Lagrangian position X, time t, or in terms of some function of X and t, such as a characteristic variable [14]. In the next section, we shall develop two different wave front expansions; they will require that we know the jump in a function both as a function of X and as a function of t, and so in this section we shall define the jump in these two ways.

Suppose that a wave front arrives at the position X at time $T(X)$. Then the jump $[f]$ in a function $f(X, t)$ is

$$[f] = f_b(X) - f_a(X), \tag{2a}$$

where

$$f_b(X) = \lim_{t \to T(X)^+} f(X, t) \tag{2b}$$

$$f_a(X) = \lim_{t \to T(X)^-} f(X, t). \tag{2c}$$

When using definition (2), we assume that the Lagrangian velocity of the wave front does not vanish for any finite interval of time, for if this were the case, X would be constant during this time interval, and this would automatically imply a constant value of $[f]$.

In (2) the jump is considered to be a function of position, but it can also be written as a function of time [3, 6, 7]. Let $N(t)$ be the Lagrangian position of the wave front at time t. Then

$$[f] = f'_b(t) - f'_a(t), \tag{3a}$$

where

$$f'_b(t) = \lim_{X \to N(t)^-} f(X, t) \tag{3b}$$

$$f'_a(t) = \lim_{X \to N(t)^+} f(X, t). \tag{3c}$$

Note that, according to definition (3), the Lagrangian velocity of the wave front may vanish for some finite interval of time without forcing $[f]$ to be constant during this interval.

4. TAYLOR EXPANSIONS

If we regard the particle velocity v, defined as

$$v = \left(\frac{\partial u}{\partial t}\right)_X, \tag{4}$$

to be a function of the two variables X and t, then the Taylor expansion about the Lagrangian position X_0 and time t_0 for the velocity is

$$v(X, t) = v(X_0, t_0) + \frac{\partial v}{\partial X}(X_0, t_0)(X - X_0) + \frac{\partial v}{\partial t}(X_0, t_0)(t - t_0) + \dots \tag{5}$$

Now it is desired to choose X_0 and t_0 so that the partial derivative terms in (5) are evaluated just behind the wave front and thus become propagating discontinuities. One way to do this is to let

$$X_0 \to X^-, \qquad t_0 \to T(X)^+. \tag{6}$$

Then, according to definition (2b), series (5) becomes the wave front expansion

$$v(X, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^k v}{\partial t^k}\right)_b (X)(t - T(X))^k. \tag{7}$$

Series (7) has been given by Reiss [13] in his investigation of one dimensional waves in inhomogeneous elastic materials.

Another way to evaluate the partial derivative terms in (5) just behind the wave front is to let

$$X_0 \to N(t)^-, \quad t_0 \to t^+. \tag{8}$$

Then, according to definition (3b), series (5) becomes the wave front expansion

$$v(X, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^k v}{\partial X^k}\right)_b (t)(X - N(t))^k. \tag{9}$$

By considering v to be a function of variables other than X and t, other series can be developed in which the partial derivative terms of the series are propagating discontinuities. For example, we could consider v to be a function of the characteristic variables of the problem [5], say α and β. Then by expanding v about α_0 and β_0 and by properly choosing α_0 and β_0, expansions other than (7) and (9) can be obtained. Such characteristic expansions are most simple for the case of propagation in linear and semilinear viscoelastic materials, where the Lagrangian first order wave front and sound speeds are constant, and where the characteristic variables can therefore be determined independently of the boundary conditions of the problem [14].

Having developed two different series which are not mutually exclusive brings up an important question; namely, given a choice of series, which one should be selected? In our work, (7) is chosen because it is easier to relate the initial values of the partial derivatives in (7) to the boundary condition than is the case with (9), and also because the series used in [1, 2, 13, 15–17] are of the same type as (7), and so (7) is most convenient for comparison purposes. A better selection criterion, however, might be convergence. If either of the series (7) or (9) should, for a particular problem, converge appreciably faster than the other, this could be an important advantage. We shall not investigate convergence in the present study, however.

5. JUMP EQUATIONS

Kinematic compatibility

As a first step in deriving governing equations for the unknown partial derivatives appearing in series (7), we shall develop some kinematic results. Thomas [18] calls the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}[f] = \left[\frac{\partial f}{\partial t}\right] + C\left[\frac{\partial f}{\partial X}\right], \tag{10a}$$

where the Lagrangian wave speed C is defined by

$$C = \frac{\mathrm{d}N}{\mathrm{d}t}, \tag{10b}$$

a kinematic condition of compatibility of the first order for the function f. A relationship which is similar to (10) but which we shall find more convenient to use is

$$C\frac{\mathrm{d}}{\mathrm{d}X}[f] = \left[\frac{\partial f}{\partial t}\right] + C\left[\frac{\partial f}{\partial X}\right]. \tag{11}$$

Some important equations which can be found from (11) and which will be used in later work will now be developed.

The compatibility condition (11) along with definition (4), the definition of engineering strain

$$\varepsilon = \left(\frac{\partial u}{\partial X}\right)_t, \tag{12}$$

and the continuity condition

$$[u] = 0, \tag{13}$$

give

$$[\varepsilon] = -\frac{1}{C}[v]. \tag{14a}$$

In addition, (11) along with (4), (12) and (13) gives

$$\left[\frac{\partial^k \varepsilon}{\partial t^k}\right] = \frac{\mathrm{d}}{\mathrm{d}X}\left[\frac{\partial^{k-1} v}{\partial t^{k-1}}\right] - \frac{1}{C}\left[\frac{\partial^k v}{\partial t^k}\right]; \qquad k = 1, 2, 3, \ldots. \tag{14b}$$

In (14) we have divided by C and so implicitly assumed that

$$C \neq 0. \tag{15}$$

Balance of momentum

Balance of momentum for a propagating discontinuity implies that [18]

$$[\sigma] = -\rho C[v] \tag{16a}$$

$$\left[\frac{\partial^k \sigma}{\partial X^k}\right] = \rho\left[\frac{\partial^k v}{\partial X^{k-1}\partial t}\right]; \qquad k = 1, 2, 3, \ldots, \tag{16b}$$

where σ is the engineering stress; that is, force per unit undisturbed cross-sectional area, and ρ is the mass density of the undisturbed material.

Some results that will be used in later work will now be derived by combining the momentum results (16) with the compatibility equation (11). Applying (11) to the engineering stress σ gives

$$\left[\frac{\partial \sigma}{\partial t}\right] = C\left(\frac{\mathrm{d}}{\mathrm{d}X}[\sigma] - \left[\frac{\partial \sigma}{\partial X}\right]\right). \tag{17}$$

Using (16a) and, for $k = 1$, (16b) in this last result gives

$$\left[\frac{\partial \sigma}{\partial t}\right] = -\rho C\left(\frac{\mathrm{d}}{\mathrm{d}X}(C[v]) + \left[\frac{\partial v}{\partial t}\right]\right). \tag{18}$$

Similarly, one can show that

$$\left[\frac{\partial^2 \sigma}{\partial t^2}\right] = -\rho C\left(\frac{\mathrm{d}}{\mathrm{d}X}\left(C\frac{\mathrm{d}}{\mathrm{d}X}(C[v])\right) + \frac{\mathrm{d}}{\mathrm{d}X}\left(C\left[\frac{\partial v}{\partial t}\right]\right) + \left[\frac{\partial^2 v}{\partial t^2}\right]\right). \tag{19}$$

Expressions for the jump in higher order time derivatives of σ should be apparent after looking at (16a), (18) and (19).

Constitutive equation

A special uniaxial isothermal constitutive equation, taken from Schapery's thermodynamic constitutive theory [10], is

$$\sigma = a_F(\varepsilon)\left(E_0\varepsilon + \int_0^t E_t(t-t')\varepsilon(X,t')\,\mathrm{d}t'\right), \tag{20a}$$

where

$$a_F = a_F(\varepsilon) = \begin{cases}\text{material property function that satisfies}\\ \text{the relations } a_F(\varepsilon) > 0 \text{ and } a_F(0) = 1\end{cases} \tag{20b}$$

$$E = E(t) = \begin{cases}\text{uniaxial relaxation modulus of linear}\\ \text{viscoelasticity}\end{cases} \tag{20c}$$

$$E_0 = E(0) \tag{20d}$$

$$E_t = \frac{\mathrm{d}E}{\mathrm{d}t}. \tag{20e}$$

By setting $a_F = 1$ in (20a), the uniaxial isothermal constitutive equation for linear viscoelasticity is obtained. Thus the function a_F, which in (20a) is associated with both the free energy of the material and the entropy production [10], acts as a strain dependent stiffening or softening factor. A great deal of relaxation and constant strain rate data for unfilled amorphous polymers, some of which can be found in [19–21], is in agreement with (20a).

Perhaps a better understanding of the role of a_F in (20a) can be obtained by actually finding a_F for a material whose isothermal equilibrium behavior is Neohookean. For such a material, the stress–strain behavior, which is illustrated in Fig. 1, is given by [22]

$$\sigma = \frac{E_e}{3}\frac{\varepsilon^2 + 3\varepsilon + 3}{(\varepsilon+1)^2}\varepsilon. \tag{21}$$

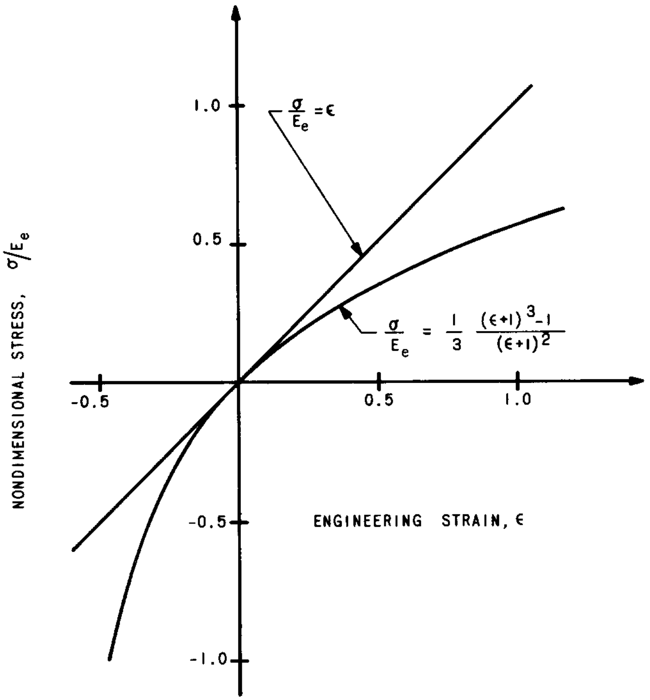

FIG. 1. Stress–strain curve for a Neohookean material.

But under isothermal equilibrium conditions, (20a) reduces to

$$\sigma = a_F(\varepsilon)E_e\varepsilon, \tag{22a}$$

where

$$E_e = E(\infty). \tag{22b}$$

So comparison of (21) and (22a) shows that

$$a_F = \frac{1}{3}\frac{\varepsilon^2+3\varepsilon+3}{(\varepsilon+1)^2}. \tag{23}$$

Jump conditions implied by (20a) are derived in the Appendix. When the material ahead of the wave front is disturbed, it is shown that

$$[\sigma] = [a_F]\left(\frac{\sigma}{a_F}\right)_a + (a_F)_b E_0[\varepsilon]. \tag{24}$$

For an initially undisturbed material, (24) reduces to

$$\sigma_b = a_F E_0 \varepsilon_b, \tag{25}$$

and it is also found that

$$\left(\frac{\partial\sigma}{\partial t}\right)_b = a_F'\left(\frac{\partial\varepsilon}{\partial t}\right)_b E_0\varepsilon_b + a_F\left(E_0\left(\frac{\partial\varepsilon}{\partial t}\right)_b + E_1\varepsilon_b\right) \tag{26}$$

$$\left(\frac{\partial^2\sigma}{\partial t^2}\right)_b = \left\{a_F'\left(\frac{\partial^2\varepsilon}{\partial t^2}\right)_b + a_F''\left(\left(\frac{\partial\varepsilon}{\partial t}\right)_b\right)^2\right\}E_0\varepsilon_b$$
$$+2a_F'\left(\frac{\partial\varepsilon}{\partial t}\right)_b\left(E_0\left(\frac{\partial\varepsilon}{\partial t}\right)_b + E_1\varepsilon_b\right) + a_F\left(E_0\left(\frac{\partial^2\varepsilon}{\partial t^2}\right)_b + E_1\left(\frac{\partial\varepsilon}{\partial t}\right)_b + E_2\varepsilon_b\right), \tag{27}$$

where

$$E_i = \left(\frac{\mathrm{d}^i E}{\mathrm{d}t^i}\right)_{t=0}; \qquad i = 0, 1, 2, \ldots, \tag{28}$$

and where primes affixed to a_F denote differentiation and a_F and its derivatives are functions of ε_b. Expressions for the jumps in higher order time derivatives of σ can be found by extending the work in the Appendix.

6. SPEED OF PROPAGATION OF FIRST ORDER AND SOUND WAVES

In order to obtain an explicit functional form for the Lagrangian speed C of a first order wave front propagating into a nonlinear material having constitutive equation (20a), we use the constitutive result (24) along with the momentum equation (16a) and the compatibility condition (14a) and find that either

$$C^2 = \frac{1}{\rho}\left\{a_F(\varepsilon_b)E_0 + \frac{[a_F]}{[\varepsilon]}\frac{\sigma_a}{a_F(\varepsilon_a)}\right\} = 0 \tag{29}$$

or

$$C^2 = \frac{1}{\rho}\left\{a_F(\varepsilon_b)E_0 + \frac{[a_F]}{[\varepsilon]}\frac{\sigma_a}{a_F(\varepsilon_a)}\right\} \neq 0. \tag{30}$$

Result (29) corresponds to a contact discontinuity for which

$$[\sigma] = 0, \qquad [v] = 0, \qquad [\varepsilon] \neq 0. \tag{31}$$

While a discontinuity such as (29), (31) may be of interest in certain situations, such as the interaction of two first order waves or the reflection-transmission of a wave passing through a junction of two materials having different densities [14], they are not needed in our present work. For further discussion of contact discontinuities in ideal fluids and elastic solids, we refer the reader to [5 and 23], respectively.

Two special cases of (30) are of particular interest. First, when the material ahead of the wave front is undisturbed, (30) reduces to

$$C^2 = a_F(\varepsilon_b)\frac{E_0}{\rho}. \tag{32}$$

Second, we wish to use (30) to find the speed of sound in nonlinear material (20a). In [5] it is shown that a sound wave can be interpreted as an infinitely weak shock. Motivated by this result, we shall define a sound wave as an infinitely weak first order wave. Then it is possible to find the Lagrangian sound speed c from (30) by letting the strength—as measured by the absolute value of $[\varepsilon]$—of the first order wave front go to zero. Such a procedure shows that

$$c^2 = \frac{1}{\rho}\left(a_F(\varepsilon)E_0 + \frac{a_F'(\varepsilon)}{a_F(\varepsilon)}\sigma\right). \tag{33}$$

Note that according to our definition of a sound wave, an nth order wave is a special case of a sound wave when $n \geq 2$. This is because all the partial time derivatives of the displacement may be continuous at the "front" of a sonic disturbance. Therefore, the Lagrangian speed C_n of an nth order wave is the same as the Lagrangian sound speed c given by (33).

7. STABILITY OF FIRST ORDER WAVE FRONTS

We shall now discuss conditions that must be satisfied in order for a first order wave front to be stable to small disturbances. The approach taken is similar to that used by Bland [24] for nonlinear elastic materials.

Let us consider an example. Suppose that the jump in the strain at a first order wave front is $[\varepsilon]$ at time t. If a small incremental wave with a strain amplitude of $d[\varepsilon]$ at its front should break away from the front, will this increment, which moves with the local Lagrangian sound speed, tend to move back toward the wave front (which would be the case for a front stable to small disturbances), or will it tend to move away from the front (which would be the case for a front not stable to small disturbances), or will it simply move at the same speed as the front (which would be the limiting case for a stable front)? It is important to realize that some authors (see [9, 23]) have suggested that stability with respect to small disturbances may be the criterion needed to insure the uniqueness of solutions to certain nonlinear wave propagation problems.

For simplicity, we shall confine the remaining discussion in this section to first order wave fronts propagating into a previously undisturbed material having constitutive equation (20a). For such a front moving in the direction of increasing X, equations (25) and (33) show that the sound speeds immediately ahead of and behind the front are, respectively,

$$c_a = \left(\frac{E_0}{\rho}\right)^{\frac{1}{2}} \tag{34}$$

$$c_b = \left\{\frac{E_0}{\rho}(a_F(\varepsilon_b) + a_F'(\varepsilon_b)\varepsilon_b)\right\}^{\frac{1}{2}}. \tag{35}$$

Table 1 summarizes the stability conclusions that can be drawn from (34), (35) and the positive square root of (32). The terminology "evolutionary", "intermediate" and "exceptional" that appears in the Table corresponds to that used by Collins in [23] for an elastic material. Note that, according to our definition, only an evolutionary type first order discontinuity is a shock front.

For the Neohookean type material discussed previously, equation (23) can be used to show that for an expansive wave front ($\varepsilon_b > 0$)

$$a_F(\varepsilon_b) < 1, \qquad a_F'(\varepsilon_b)\varepsilon_b < 0. \tag{36}$$

TABLE 1. STABILITY OF A FIRST ORDER WAVE FRONT MOVING INTO A PREVIOUSLY UNDISTURBED NONLINEAR MATERIAL

	$a_F'(\varepsilon_b)\varepsilon_b < 0$	$a_F'(\varepsilon_b)\varepsilon_b = 0$	$a_F'(\varepsilon_b)\varepsilon_b > 0$
$a_F(\varepsilon_b) < 1$	$c_b < C < c_a$ Unstable	$c_b = C < c_a$ Unstable	$C < c_a, c_b$ Unstable
$a_F(\varepsilon_b) = 1$	$c_b < C = c_a$ Unstable	$c_b = C = c_a$ Stable (exceptional)	$c_a = C < c_b$ Stable (intermediate)
$a_F(\varepsilon_b) > 1$	$c_b, c_a < C$ Unstable	$c_a < C = c_b$ Stable (intermediate)	$c_a < C < c_b$ Stable (evolutionary)

So according to Table 1, an expansive wave front is unstable. It will therefore smooth out immediately and propagate with the sonic velocity given by (34). On the other hand, for a compressive wave front ($\varepsilon_b < 0$), it can be shown from (23) that

$$a_F(\varepsilon_b) > 1, \qquad a_F'(\varepsilon_b)\varepsilon_b > 0. \tag{37}$$

So according to Table 1, a compressive wave is a genuine shock front. Therefore it propagates with a speed given by (32).

8. THE SHOCK SOLUTION

We shall assume that the particle velocity at the end of an initially undisturbed rod is specified in the form

$$v(0,t) = \begin{cases} 0, & t < 0 \\ \sum_{k=0}^{\infty} \frac{1}{k!} v_k t^k, & t \geq 0. \end{cases} \tag{38}$$

Also assume that the conditions (37) are satisfied for all times under consideration. Then, according to Table 1, the wave front is stable to small disturbances and is a true shock; therefore the particle velocity for some distance behind the wave front can be represented by the expansion (7).

The time of arrival of the wave front at the Lagrangian coordinate X is given by

$$T(X) = \int_0^X \frac{\mathrm{d}X'}{C(v_b(X'))}. \tag{39}$$

In (39) we assume that we can find the wave speed C as a function of v_b by first substituting (14a) into (32) to obtain

$$C = \left(a_F(-v_b/C) \frac{E_0}{\rho} \right)^{\frac{1}{2}} \tag{40}$$

and then solving this last equation for $C(v_b)$. It is interesting to note that when result (23) for a Neohookean type material is substituted into (40), we can show, after solving a cubic equation [14], that

$$\frac{C}{C_0} = \frac{1}{2}\frac{v_b}{C_0} + \left\{ \frac{1}{2} + \frac{1}{4}\left(\frac{v_b}{C_0}\right)^2 + \frac{1}{2}\left(1 + \frac{4}{3}\left(\frac{v_b}{C_0}\right)^2\right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}, \tag{41a}$$

where

$$C_0 = \left(\frac{E_0}{\rho_0}\right)^{\frac{1}{2}}. \tag{41b}$$

A graph of result (41a) is shown in Fig. 2. Note that (41a) is valid irrespective of whether the material is elastic or viscoelastic.

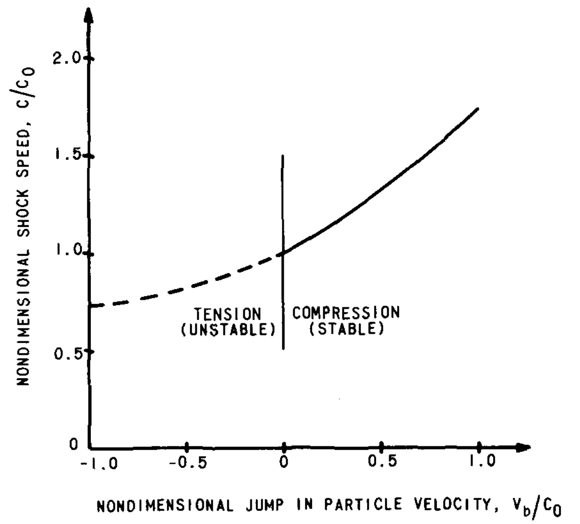

FIG. 2. Shock speed vs. jump in particle velocity for a Neohookean type material.

In order to formally solve the wave propagation problem that we have posed, we must solve for the partial derivative terms appearing in (7). This shall now be done by combining previously developed compatibility, momentum and constitutive results.

Substituting compatibility results (14a), and, for $k = 1$ and $k = 2$, (14b), and momentum–compatibility results (18) and (19) into constitutive results (26) and (27) gives

$$\frac{a_F' E_0}{C^2} v_b \left(\frac{\partial v}{\partial t}\right)_b = -\frac{a_F E_0}{C}\frac{\mathrm{d}}{\mathrm{d}X}(Cv_b) + \frac{E_0 v_b}{C} a_F' \frac{\mathrm{d}}{\mathrm{d}X} v_b - a_F\left(E_0 \frac{\mathrm{d}}{\mathrm{d}X} v_b - E_1 \frac{v_b}{C}\right) \tag{42}$$

$$\begin{aligned} \frac{a_F' E_0}{C^2} v_b \left(\frac{\partial^2 v}{\partial t^2}\right)_b = & -\frac{a_F E_0}{C}\left\{\frac{\mathrm{d}}{\mathrm{d}X}\left(C\frac{\mathrm{d}}{\mathrm{d}X}(Cv_b)\right) + \frac{\mathrm{d}}{\mathrm{d}X}\left(C\left(\frac{\partial v}{\partial t}\right)_b\right)\right\} \\ & + \frac{E_0 v_b}{C}\left\{a_F' \frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\partial v}{\partial t}\right)_b + a_F''\left(\frac{\mathrm{d}}{\mathrm{d}X} v_b - \frac{1}{C}\left(\frac{\partial v}{\partial t}\right)_b\right)^2\right\} \\ & - 2a_F'\left(\frac{\mathrm{d}}{\mathrm{d}X} v_b - \frac{1}{C}\left(\frac{\partial v}{\partial t}\right)_b\right)\left\{E_0\left(\frac{\mathrm{d}}{\mathrm{d}X} v_b - \frac{1}{C}\left(\frac{\partial v}{\partial t}\right)_b\right) - E_1\frac{v_b}{C}\right\} \\ & - a_F\left\{E_0 \frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\partial v}{\partial t}\right)_b + E_1\left(\frac{\mathrm{d}}{\mathrm{d}X} v_b - \frac{1}{C}\left(\frac{\partial v}{\partial t}\right)_b\right) - E_2\frac{v_b}{C}\right\}, \end{aligned} \tag{43}$$

where we have also used result (32), and where a_F and its derivatives are all functions of $-v_b/C$.

Now we have seen from (40) that the wave speed C can be regarded as a function of v_b, and so equation (42) can be written as

$$\left(\frac{\partial v}{\partial t}\right)_b = f_1(v_b)\frac{\mathrm{d}}{\mathrm{d}X}v_b + g_1(v_b), \tag{44a}$$

where

$$f_1(v_b) = C\left(1 - \frac{a_F}{a'_F}\left(2\frac{C}{v_b} + \frac{\mathrm{d}C}{\mathrm{d}v_b}\right)\right) \tag{44b}$$

$$g_1(v_b) = C\frac{a_F}{a'_F}\frac{E_1}{E_0}. \tag{44c}$$

Similarly, substituting (42) into (43) gives a result having the form

$$\left(\frac{\partial^2 v}{\partial t^2}\right)_b = f_2(v_b)\frac{\mathrm{d}^2}{\mathrm{d}X^2}v_b + g_2\left(v_b, \frac{\mathrm{d}}{\mathrm{d}X}v_b\right). \tag{45}$$

By extending the results of Section 5 and combining them so as to extend the results (42) and (43) to obtain expressions for higher order time derivatives of the particle velocity, it can be shown that

$$\left(\frac{\partial^k v}{\partial t^k}\right)_b = f_k(v_b)\frac{\mathrm{d}^k}{\mathrm{d}X^k}v_b + g_k\left(v_b, \frac{\mathrm{d}}{\mathrm{d}X}v_b, \ldots, \frac{\mathrm{d}^{k-1}}{\mathrm{d}X^{k-1}}v_b\right); \qquad k = 1, 2, 3, \ldots. \tag{46}$$

The functions f_k and g_k appearing in (46) will not be solved for when $k \geq 2$, but since they could be, they are considered as known.

Our next task is to use the results we have just developed to find the constant coefficients V_k appearing in the expansion

$$v_b(X) = \sum_{k=0}^{\infty} \frac{1}{k!} V_k X^k, \tag{47a}$$

where

$$V_k = \left(\frac{\mathrm{d}^k}{\mathrm{d}X^k}v_b(X)\right)_{X=0}. \tag{47b}$$

The particle velocity $v_b(X)$ just behind the wave front will then be known, at least formally, and so we shall be able to use (46) to find the partial derivatives $(\partial^k v/\partial t^k)_b(X)$ and (39) to find the time of arrival $T(X)$, and we shall then have solved our wave propagation problem as it was formulated.

Let us first find V_0. Using result (47b) for $k = 0$ gives the result that

$$V_0 = v_0, \tag{48}$$

where v_0 is defined by boundary condition (38).

Next we find V_k when $k \geq 1$. Evaluating both sides of (46) at $X = 0$ and using (47b) and (48), it is not hard to show that

$$V_k = \frac{v_k - g_k(v_0, V_1, \ldots, V_{k-1})}{f_k(v_0)}; \qquad k = 1, 2, 3, \ldots, \tag{49}$$

where v_k is defined by boundary condition (38). So from (49) we can successively solve for $V_1, V_2, V_3, \ldots$.

An interesting special case of shock propagation occurs when C is a positive constant. From (40) we see that v_b is also constant, and equations (38), either (42) or (44) and (43) give the respective results

$$v_b = v_0 \tag{50}$$

$$\left(\frac{\partial v}{\partial t}\right)_b = \frac{a_F E_1}{a_F' E_0} C \tag{51}$$

$$\left(\frac{\partial^2 v}{\partial t^2}\right)_b = \left(\frac{a_F''}{a_F'}\frac{1}{C} - \frac{2}{v_0}\right)\left(\frac{a_F E_1}{a_F' E_0} C\right)^2 - 2\frac{E_1}{E_0}\left(\frac{a_F E_1}{a_F' E_0}\right) C + \frac{a_F E_1}{a_F' E_0 v_0} C + \frac{a_F E_2}{a_F' E_0} C, \tag{52}$$

where a_F and its derivatives are functions of $-v_0/C$.

These last three equations show that the first three partial derivative terms in Taylor expansion (7) are constant, and from (46) we can see that this is also true for the jumps in all of the higher order partial derivatives. Since v_b is constant, time of arrival (39) becomes

$$T(X) = \frac{X}{C}, \tag{53}$$

and expansion (7) can be written as

$$v(X, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^k v}{\partial t^k}\right)_b \left(t - \frac{X}{C}\right)^k, \tag{54}$$

where all of the partial derivative terms in (54) are constants which depend upon the value v_0. Thus the wave form travels rigidly down the rod.

Let us now find the boundary conditions that will generate a rigid wave form. Since all of the partial derivative terms in (54) must satisfy the initial conditions

$$\left(\left(\frac{\partial^k v}{\partial t^k}\right)_b\right)_{t=0} = v_k; \qquad k = 0, 1, 2, \ldots, \tag{55}$$

where the constants v_k are defined by (38), the velocity that must be applied to the end of the rod to maintain a constant speed of propagation is

$$v(0, t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\partial^k v}{\partial t^k}\right)_b t^k, \tag{56}$$

where, according to (50), the first partial derivative in (56) is v_0, and the second and third partial derivatives are given in terms of v_0 by (51) and (52), respectively.

For an elastic material, the functions E_i defined by (28) vanish for $i = 1, 2, 3, \ldots$, and so the constant shock speed solution (54) becomes

$$v(X, t) = v_0, \tag{57}$$

where v_0 is arbitrary.

Before concluding this section, let us note that the shock solution (7) does not directly reduce to the solution for an acceleration wave in a nonlinear material, nor does it reduce to the solution for a first order wave in a linear material. To see this, write (44) as

$$\left(\frac{\partial v}{\partial t}\right)_b = C\left(1-\frac{a_F}{a'_F}\left(2\frac{C}{v_b}+\frac{\mathrm{d}C}{\mathrm{d}v_b}\right)\right)\frac{\mathrm{d}v_b}{\mathrm{d}X}+C\frac{a_F}{a'_F}\frac{E_1}{E_0}. \tag{58}$$

Now by letting $v_b(X)$ go to zero, the acceleration just behind the front becomes indeterminate, so expansion (7) for a shock wave does not directly reduce to an expansion for an acceleration wave. Also, as a'_F goes to zero, the acceleration just behind the front becomes infinite, so expansion (7) does not directly reduce to an expansion for a first order wave in a linear material.

9. DISCUSSION OF THE SOLUTION

Although the series solution developed in the last section gives a good deal of physical insight into wave propagation and leads to some important conclusions, it does have certain limitations. For example, if a second shock wave should form somewhere between the original one and the end of the rod, the largest region for which a wave front series solution can be valid is between the two shock fronts. Also, a wave front expansion cannot be used after a first order wave front becomes unstable to small disturbances. In these two instances, however, one can obtain a numerical solution to the wave propagation problem by using the method of characteristics [14].

As can be seen by looking at equations (42) and (43), a wave front expansion depends on knowing the initial values of the relaxation modulus and its derivatives. Although Sackman and Kaya [25] have pointed out that it is practically impossible to experimentally obtain more than the first two or three of these values, the use of the time–temperature superposition principle [26] and extrapolation methods—such as assuming that a particular functional form for the relaxation modulus is valid from time $t = 0$ to as large a time as is necessary—should overcome this difficulty to a considerable extent.

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APPENDIX

We wish to find the jump conditions implied by constitutive equation (20a).

First, use the product rule of differentiation and the fact that a jump in the sum of n functions is equal to the sum of the jumps of each of the n functions to write

$$[\sigma] = \left[a_F\left(\frac{\sigma}{a_F}\right)\right] \tag{A1}$$

$$\left[\frac{\partial\sigma}{\partial t}\right] = \left[\frac{\partial a_F}{\partial t}\left(\frac{\sigma}{a_F}\right)\right] + \left[a_F\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right] \tag{A2}$$

$$\left[\frac{\partial^2\sigma}{\partial t^2}\right] = \left[\frac{\partial^2 a_F}{\partial t^2}\left(\frac{\sigma}{a_F}\right)\right] + 2\left[\frac{\partial a_F}{\partial t}\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right] + \left[a_F\frac{\partial^2}{\partial t^2}\left(\frac{\sigma}{a_F}\right)\right]. \tag{A3}$$

Now from either (2) or (3) or any other consistent definition of the jump, it is not hard to show that

$$[fg] = [f]g_a + f_b[g]. \tag{A4}$$

So using (A4) in (A1)–(A3) gives the respective results

$$[\sigma] = [a_F]\left(\frac{\sigma}{a_F}\right)_a + (a_F)_b\left[\frac{\sigma}{a_F}\right] \tag{A5}$$

$$\left[\frac{\partial\sigma}{\partial t}\right] = \left[\frac{\partial a_F}{\partial t}\right]\left(\frac{\sigma}{a_F}\right)_a + \left(\frac{\partial a_F}{\partial t}\right)_b\left[\frac{\sigma}{a_F}\right] + [a_F]\left(\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right)_a + (a_F)_b\left[\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right] \tag{A6}$$

$$\begin{aligned}\left[\frac{\partial^2\sigma}{\partial t^2}\right] = {} & \left[\frac{\partial^2 a_F}{\partial t^2}\right]\left(\frac{\sigma}{a_F}\right)_a + \left(\frac{\partial^2 a_F}{\partial t^2}\right)_b\left[\frac{\sigma}{a_F}\right] + 2\left[\frac{\partial a_F}{\partial t}\right]\left(\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right)_a \\ & + 2\left(\frac{\partial a_F}{\partial t}\right)_b\left[\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right] + [a_F]\left(\frac{\partial^2}{\partial t^2}\left(\frac{\sigma}{a_F}\right)\right)_a + (a_F)_b\left[\frac{\partial^2}{\partial t^2}\left(\frac{\sigma}{a_F}\right)\right].\end{aligned} \tag{A7}$$

Now in [10] it is shown that $a_F(\varepsilon)$ is always positive. Therefore, constitutive equation (20a) can be written as

$$\frac{\sigma}{a_F} = E_0\varepsilon + \int_0^t E_t(t-t')\varepsilon(X,t')\,\mathrm{d}t'. \tag{A8}$$

Repeated differentiation of (A8) gives

$$\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right) = E_0\frac{\partial\varepsilon}{\partial t} + E_1\varepsilon + \int_0^t E_{tt}(t-t')\varepsilon(X,t')\,\mathrm{d}t' \tag{A9}$$

$$\frac{\partial^2}{\partial t^2}\left(\frac{\sigma}{a_F}\right) = E_0\frac{\partial^2\varepsilon}{\partial t^2} + E_1\frac{\partial\varepsilon}{\partial t} + E_2\varepsilon + \int_0^t E_{ttt}(t-t')\varepsilon(X,t')\,\mathrm{d}t', \tag{A10}$$

where

$$E_i = \left(\frac{\mathrm{d}^i E}{\mathrm{d}t^i}\right)_{t=0}; \qquad i = 0, 1, 2, \ldots, \tag{A11}$$

and where each subscript t affixed to the relaxation modulus E denotes a time differentiation.

To find the jump in σ/a_F as a function of the jump in strain, use (3) and (A8) to write

$$\left[\frac{\sigma}{a_F}\right] = E_0\varepsilon_b(t) + \int_0^{t_b} E_t(t_b-t')\varepsilon(X,t')\,\mathrm{d}t' - E_0\varepsilon_a(t) - \int_0^{t_a} E_t(t_a-t')\varepsilon(X,t')\,\mathrm{d}t'. \tag{A12}$$

But, since $t_a = t_b$, equation (A12) in conjunction with (3) shows that

$$\left[\frac{\sigma}{a_F}\right] = E_0[\varepsilon]. \tag{A13}$$

Similarly, the fact that $t_a = t_b$ along with definition (3) and equations (A9) and (A10) can be used to show that

$$\left[\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right] = E_0\left[\frac{\partial\varepsilon}{\partial t}\right] + E_1[\varepsilon] \tag{A14}$$

$$\left[\frac{\partial^2}{\partial t^2}\left(\frac{\sigma}{a_F}\right)\right] = E_0\left[\frac{\partial^2\varepsilon}{\partial t^2}\right] + E_1\left[\frac{\partial\varepsilon}{\partial t}\right] + E_2[\varepsilon]. \tag{A15}$$

Substituting (A13)–(A15) into (A5)–(A7) and using the chain rule to carry out the derivatives of a_F gives the respective results

$$[\sigma] = [a_F]\left(\frac{\sigma}{a_F}\right)_a + (a_F)_b E_0[\varepsilon] \tag{A16}$$

$$\left[\frac{\partial\sigma}{\partial t}\right] = \left[a_F'\frac{\partial\varepsilon}{\partial t}\right]\left(\frac{\sigma}{a_F}\right)_a + \left(a_F'\frac{\partial\varepsilon}{\partial t}\right)_b E_0[\varepsilon] + [a_F]\left(\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right)_a + (a_F)_b\left(E_0\left[\frac{\partial\varepsilon}{\partial t}\right] + E_1[\varepsilon]\right) \tag{A17}$$

$$\begin{aligned}\left[\frac{\partial^2\sigma}{\partial t^2}\right] = {} & \left[a_F''\left(\frac{\partial\varepsilon}{\partial t}\right)^2 + a_F'\frac{\partial^2\varepsilon}{\partial t^2}\right]\left(\frac{\sigma}{a_F}\right)_a + \left(a_F''\left(\frac{\partial\varepsilon}{\partial t}\right)^2 + a_F'\frac{\partial^2\varepsilon}{\partial t^2}\right)_b E_0[\varepsilon] + 2\left[a_F'\frac{\partial\varepsilon}{\partial t}\right]\left(\frac{\partial}{\partial t}\left(\frac{\sigma}{a_F}\right)\right)_a \\ & + 2\left(a_F'\frac{\partial\varepsilon}{\partial t}\right)_b\left(E_0\left[\frac{\partial\varepsilon}{\partial t}\right] + E_1[\varepsilon]\right) + [a_F]\left(\frac{\partial^2}{\partial t^2}\left(\frac{\sigma}{a_F}\right)\right)_a + (a_F)_b\left(E_0\left[\frac{\partial^2\varepsilon}{\partial t^2}\right] + E_1\left[\frac{\partial\varepsilon}{\partial t}\right] \right. \\ & \left. + E_2[\varepsilon]\right).\end{aligned} \tag{A18}$$

For the particular case when the material ahead of the wave front is undisturbed, all the terms having a subscript a vanish and equations (A16)–(A18) become, respectively,

$$\sigma_b = (a_F)_b E_0 \varepsilon_b \tag{A19}$$

$$\left(\frac{\partial\sigma}{\partial t}\right)_b = \left(a_F'\frac{\partial\varepsilon}{\partial t}\right)_b E_0\varepsilon_b + (a_F)_b\left(E_0\left(\frac{\partial\varepsilon}{\partial t}\right)_b + E_1\varepsilon_b\right) \tag{A20}$$

$$\begin{aligned}\left(\frac{\partial^2\sigma}{\partial t^2}\right)_b = {} & \left(a_F''\left(\frac{\partial\varepsilon}{\partial t}\right)^2 + a_F'\frac{\partial^2\varepsilon}{\partial t^2}\right)_b E_0\varepsilon_b + 2\left(a_F'\frac{\partial\varepsilon}{\partial t}\right)_b\left(E_0\left(\frac{\partial\varepsilon}{\partial t}\right)_b + E_1\varepsilon_b\right) \\ & + (a_F)_b\left(E_0\left(\frac{\partial^2\varepsilon}{\partial t^2}\right)_b + E_1\left(\frac{\partial\varepsilon}{\partial t}\right)_b + E_2\varepsilon_b\right).\end{aligned} \tag{A21}$$

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Абстракт—Исслекуется распространение ударных волн в полубесконечных вязкоупругих стержнях. Частичная скорость за фронтом волны разлагается в ряд Тейлора вокруг фронта. Обсуждаются условия, при которых фронт волны устойчив, в случае малых возмущений. Для устойчивого фронта волны показано способ использования теории распространения поверхностей разрыва, с целью определения членов с частными производными и времени функции возвращения, которые появляются в рядах, когда материал перед фронтом волны не подвергается возмущениям. Ряд справедлив так долго, как долго фронт волны первого порядка устойчив, и так долго, когда другие ударные волны не образуются, кроме одной первоначальной. Для иллюстрации некоторых случаев теории, используется вязкоупругий материал, проявляющий неогуковое напряженно-деформированное поведение в состояниях изотермического равновесия.